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Correlation amplitude for the XXZ spin chain in the disordered regime

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Abstract

We proposed an analytical expression for the amplitude defining the long distance asymptotic of the correlation function $\langle\, \sigma^z_k \sigma^z_{k+n} \,\rangle$.

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One of the most famous model for 1D magnetic is the XXZ spin chain,

$$\mathbf{H}_{XXZ} = -\frac{J}{2} \sum_{k=-\infty}^{\infty} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \left(\sigma_k^z \sigma_{k+1}^z - 1 \right) \right) , \tag{1}$$

where σ_k^x, σ_k^y and σ_k^z are the Pauli matrices associated with the site k. The energy spectrum of the model can be studied by means of the Bethe ansatz technique (see e.g. Ref.[1] for a review). An exact calculation of correlation functions is a much challenging problem [2,3]. In the disordered regime[1]

$$-1 \leq \Delta < 1 , \qquad J > 0 , \tag{2}$$

the continuous limit of the chain (1) is described by the simple Conformal Field Theory model (the Gaussian model) and the qualitative analysis of the correlation functions are obtained by the Luther-Pershel bosonization procedure [4]. The simplest zero-temperature, equal-time correlators have the following leading behavior [4],

$$\begin{aligned} \langle \, \sigma_k^x \sigma_{k+n}^x \, \rangle &= F \; n^{-\eta} + \ldots \; , \\ \langle \, \sigma_k^z \sigma_{k+n}^z \, \rangle &= -\frac{1}{\pi^2 \, \eta} \; n^{-2} + (-1)^n \; A \; n^{-\frac{1}{\eta}} + \ldots \quad \text{as} \;\; n \to \infty \; , \end{aligned} \tag{3}$$

where dots stand for subleading terms of the asymptotics. The parameter $0 < \eta < 1$ in (3) is related with the anisotropy Δ,

$$\Delta = \cos(\pi \eta) \; . \tag{4}$$

The Luther-Pershel approach fails to predict the value of the correlation amplitudes F and A. Recent field-theoretical results [5,6] made it possible to determine the amplitude F [5],

$$\begin{aligned} F = & \frac{1}{2\,(1-\eta)^2} \left[\frac{\Gamma\big(\frac{\eta}{2-2\eta}\big)}{2\sqrt{\pi}\;\Gamma\big(\frac{1}{2-2\eta}\big)} \right]^{\eta} \times \\ & \exp\Big\{ -\int_0^{\infty} \frac{dt}{t} \left(\frac{\sinh\big(\eta t\big)}{\sinh(t) \cosh\big((1-\eta)t\big)} - \eta \, e^{-2t} \right) \Big\} \; . \end{aligned} \tag{5}$$

This expression was confirmed numerically in Ref.[7].

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[1] The substitution $J \to -J$, $\Delta \to -\Delta$ transform (1) to the unitary equivalent model. In particular, the chain with $J > 0$, $\Delta = -1$ is unitary equivalent to the $SU(2)$ invariant antiferromagnetic spin chain.

Up to now, only few analytical results were known about the second amplitude A, namely, its values at the "free fermion" point, $\Delta = 0$ [8,9],

$$A\Big|_{\Delta=0} = \frac{2}{\pi^2} = 0.2026... \ , \tag{6}$$

and at $\Delta = -\frac{1}{\sqrt{2}}$ [10],

$$A\Big|_{\Delta=-\frac{1}{\sqrt{2}}} = \Big(\frac{2}{3}\Big)^{\frac{5}{3}} \ \frac{4}{\Gamma^4\big(\frac{2}{3}\big)} = 0.6053... \ . \tag{7}$$

In this letter we propose the exact formula for the amplitude A,

$$\begin{aligned} A = & \frac{8}{\pi^2}\ \Bigg[\frac{\Gamma\big(\frac{\eta}{2-2\eta}\big)}{2\sqrt{\pi}\ \Gamma\big(\frac{1}{2-2\eta}\big)}\Bigg]^{\frac{1}{\eta}} \times \\ & \exp\Bigg\{\int_0^\infty \frac{dt}{t}\ \Bigg(\frac{\sinh\big((2\eta-1)t\big)}{\sinh(\eta t)\cosh\big((1-\eta)t\big)} - \frac{2\eta-1}{\eta}\ e^{-2t}\Bigg)\Bigg\} \ . \end{aligned} \tag{8}$$

This function satisfies Eqs.(6),(7). One can also check the behavior of A in the limit $\Delta \to -1$,

$$A\Big|_{\Delta\to-1} \to \frac{2}{\pi}\ \Big(\frac{2}{1+\Delta}\Big)^{\frac{1}{4}} \ . \tag{9}$$

The divergence is due to the irrelevant operator with the scale dimension $2\,\eta^{-1}$ occurring in the low-energy effective Hamiltonian of the spin chain [11]. For the same reason Eq.(3) defines the leading asymptotics of the correlators only for

$$\log(n) \gg \frac{1}{2-2\,\eta} \ .$$

If $\Delta = -1$, the domain of validity of (3) disappears completely. In order to examine the asymptotic in the vicinity $\Delta = -1$, we should perform the standard renormalization group resummation (see e.g. [12,13]). Using (9), one can obtain,

$$\langle\, \sigma^z_k \sigma^z_{k+n}\, \rangle\big|_{\Delta\to-1} = (-1)^n\ \sqrt{\frac{2}{\pi^3}}\ \frac{2\,\sqrt{-g_\perp}}{n\,(g_\parallel - g_\perp)}\ \big(1 + O(g)\big) \ , \tag{10}$$

with

$$g_\parallel = 2\,(1-\eta)\ \frac{1+q}{1-q} \ , \qquad g_\perp = -4\,(1-\eta)\ \frac{q^{\frac{1}{2}}}{1-q} \ . \tag{11}$$

Here $q = q(n,\eta)$ is the solution of the equation

$$q\ (1-q)^{\frac{2}{\eta}-2} = \Bigg[\frac{e^{-\gamma-1}\,\eta\,\Gamma\big(\frac{\eta}{2-2\eta}\big)}{2\,\sqrt{\pi}\,n\,\Gamma\big(\frac{1}{2-2\eta}\big)}\Bigg]^{\frac{4}{\eta}-4} \tag{12}$$

and $\gamma = 0.5772...$ is the Euler constant. Now we take the limit $g_\perp \to -g_\parallel$, corresponding to $\Delta \to -1$. The final result reproduce the prediction from [12,13] for $SU(2)$-invariant antiferromagnetic spin chain. It supports the limiting behavior (9).

The amplitude (8) was also checked against available numerical data. In Table and Figure the numerics from Ref.[7] are compared against (8). Notice that the fitting procedure [7] of the numerical data was based on formulas similar to (3)[2]. As mentioned above, the asymptotics (3) are applicable for very large n only provided $\Delta \simeq -1$. For $\Delta = -0.9$ the term $\propto n^{2-3/\eta}$, omitted in (3), gives rise a 18% correction to the value of the correlator $\langle\, \sigma^z_k \sigma^z_{k+n}\,\rangle$ at $n = 100$. At the same time the total length of the chain in [7] was 200 sites. Therefore, the discrepancy between (8) and the numerical data in the vicinity $\Delta = -1$ does not seem to contradict our conjecture seriously.

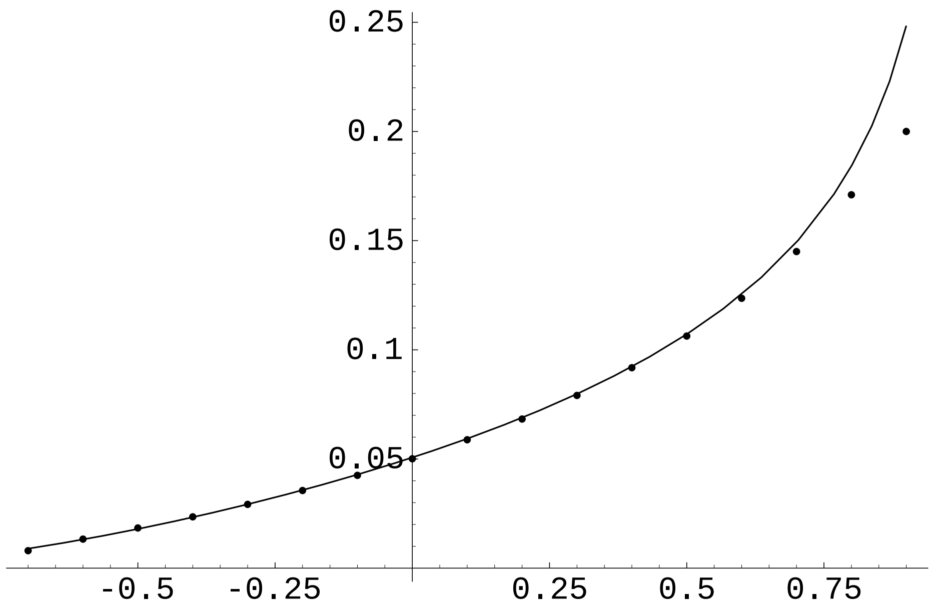

Figure. The correlation amplitude $A/4$ from (8) (vertical axis) as a function of the anisotropy parameter $-\Delta$ (horizontal axis). The bullets (see Table) were obtained in [7].

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[2] The authors adapted (3) for the spin chain with open boundaries.

$-\Delta$	$A_{num}/4$	$A/4$
-0.7	0.008(1)	0.00893
-0.6	0.0133(1)	0.01314
-0.5	0.0184(4)	0.01795
-0.4	0.0235(2)	0.02332
-0.3	0.02921(3)	0.02924
-0.2	0.03556(3)	0.03574
-0.1	0.0425(2)	0.04285
-0.0	0.0501(5)	0.05066
0.1	0.0588(3)	0.05929
0.2	0.0683(6)	0.06891
0.3	0.0791(8)	0.07978
0.4	0.0918(9)	0.09231
0.5	0.1063(9)	0.10713
0.6	0.1236(5)	0.12539
0.7	0.145(1)	0.14930
0.8	0.171(5)	0.18414
0.9	0.20(1)	0.24844

Table. The correlation amplitude $A_{num}/4$ was estimated in the paper [7]. $A/4$ follows from Eq.(8)

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